Name of College - S. S. college J-Bad
Deptt - Mathematics
Topic - Mean value theorem
Class - B.Sc II (HONS)
Time - 7.30 A.M to 9.00 A.M
Date - 24-04-2021
Teacher's Name - Dr. Shree Nath Sh
(Associate Prof)

using T01



# Lagrange's Mean value theorem
or
# First Mean value theorem

**Statement** :→ If $f(x)$ be a function such that
(i) $f(x)$ is continuous in $[a,b]$
(ii) $f(x)$ is differentiable in $(a,b)$
then there exist at least one point $c$ in $(a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

**Proof** :→ Let us consider a function $g$ defined by

$g(x) = f(x) + Ax$
where $A$ is constant to be determined such that

$$g(a) = g(b)$$

$\Rightarrow f(a) + Aa = f(b) + Ab$

$\Rightarrow f(a) - f(b) = Ab - Aa$

$\Rightarrow -[f(b) - f(a)] = A(b-a)$

$\Rightarrow -A = \dfrac{f(b)-f(a)}{b-a}$ ——— (1)

Since $g(x)$ is sum of two functions $f(x)$ and $Ax$ both of which are continuous in $[a,b]$ and also differentiable in $(a,b)$

thus $g(x)$ is continuous in $[a,b]$
$g(x)$ is differentiable in $(a,b)$
$g(a) = g(b)$

$\Rightarrow$ By Rolle's theorem there exists $c \in (a,b)$ such that $g'(c) = 0$

But $g(x) = f(x) + A(x)$

$\therefore g'(x) = f'(x) + A$

$\Rightarrow g'(c) = f'(c) + A$ $\quad = -A + g'(c) = 0$

$\therefore f'(c) = g'(c) - A$ $\quad$ since $g'(c) = 0$

$\Rightarrow -A = f'(c)$

using (1)

$$f'(c) = A = \frac{f(b) - f(a)}{b - a}$$

**Note** :- (i) The differentiability is not Required at the end points $a$ and $b$

(ii) If in the statement of mean value theorem, $b$ is replaced by $a+h$, then $c \in (a,b)$ may be written $c = a+\theta h$ for some number $0<\theta<1$

Thus $\frac{f(a+h) - f(a)}{a+h-a} = f'(a+\theta h)$

$\Rightarrow$ $f(a+h) = f(a) + h f'(a+\theta h)$ where $0<\theta<1$

. Thus the Lagrange's mean value theorem may also be stated as

If $f$ is a real valued function defined on $[a, a+h]$ is

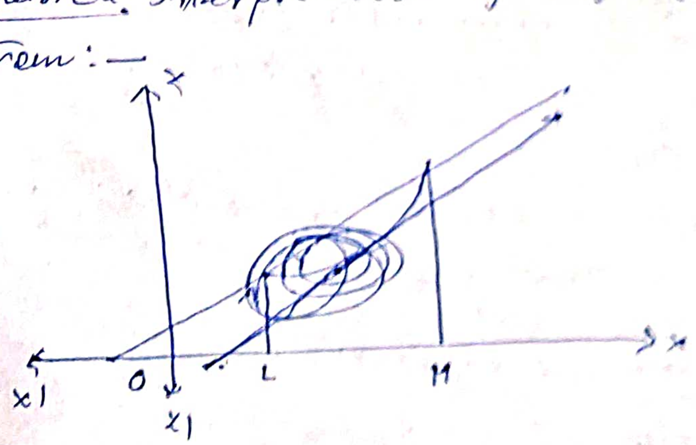
(i) continuous on $[a, a+h]$

(ii) differentiable on $(a, a+h)$

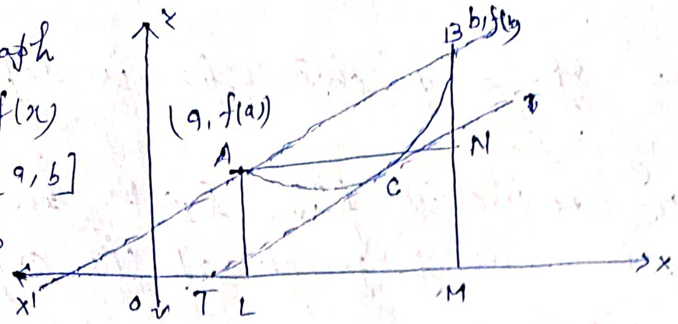
Then there Exists a number $\theta$ where $0<\theta<1$ such that

$$f(a+h) = f(a) + h f'(a+\theta h)$$
$$\theta \in (0,1)$$

**Geometrical Interpretation of Lagrange's Mean value theorem :-**



Here Graph of the $f(x)$ in $[a, b]$ is ACB




and let $a$, $c$, $b$ be the x-co-ordinates of the point A, C, B respectively on the curve $y = f(x)$

Such that the relation
$f(b) - f(a) = (b-a) f'(c)$ is satisfied

Draw AB, $BM \perp OX$
and $AN \perp BM$
Let CT be the Tangent at C

then $\dfrac{f(b) - f(a)}{b-a} = \dfrac{BM - AL}{LM} = \dfrac{BN}{AN} = \tan \angle BAN$

Since $f'(c) = \tan \angle CTX$
i.e $\angle BAN = \angle CTX$

$\Rightarrow AB \parallel CT$

Hence we have the following geometrical interpretation of Lagrange's Mean value theorem

If the Graph ACB is Continuous curve having every where a Tangent (Except possible at A and B), there must be at least one point intermediate between A and B at which the Tangent is parallel to the chord AB.

Generalise of Mean value theorem
or
Taylor's theorem.

## Taylor's Theorem

**Statement :→** If a function $f(x)$ is defined in such a way that:

(i) the $(n-1)$th derivative $f^{n-1}(x)$ exists and is continuous at every point of the closed interval $a \le x \le a+h$.

(ii) the $n$th derivative $f^{n}(x)$ exists at every point of the open interval $a < x < a+h$.

and (iii) $p$ is any positive integer, then there exists at least one number $\theta$ lying between 0 and 1 such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{(n-1)}(a)$$

$$+ \frac{h^n}{\lfloor n-1} \cdot \frac{(1-\theta)^{n-p}}{p} f^{n}(a+\theta h)$$

**Proof :→** Let us consider $g(x)$ such that

$$g(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{\lfloor 2} f''(x) + \cdots$$

$$\cdots + \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^{n-1}(x) + A(a+h-x)^p \quad \text{--- (i)}$$

Where the constant $A$ be such that

$$g(a) = g(a+h) \quad \text{--- (ii)}$$

putting $x = a$ in (i)

$$g(a) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + Ah^p \quad \text{--- (iii)}$$

Again putting $x = a+h$ in (i)

we get $g(a+h) = f(a+h)$

putting these two values in (ii)

$$f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\cdots+\frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a)+Ah^p=f(a+h) \quad \text{(iii)}$$

it gives that

$f^{n-1}(x)$ is continuous in $a\le x\le a+h$.

Therefore $f(x), f'(x); f''(x); \cdots f^{n-2}(x)$ must be continuous in

$a\le x\le a+h$.

Hence from (i)

$g(x)$ is continuous in $a\le x\le a+h$.

Again $g'(x)$ exists in $a<x<a+h$

Also $g(a)=g(a+h)$

We see that $g(x)$ satisfies all the conditions of Rolle's theorem,

thus by Rolle's theorem there must exists at least one number $\theta\in(0,1)$ such that

$$g'(a+\theta h)=0 \quad \text{(iv)}$$

Diff. (i) with respect to $x$, we get

$$g'(x)=f'(x)-f'(x)+(a+h-x)f''(x)$$
$$-(a+h-x)f''(x)+\frac{(a+h-x)^2}{\lfloor 2}f'''(x)+\cdots$$
$$\cdots+\frac{(a+h-x)^{n-2}}{\lfloor n-2}f^{n-1}(x)+\frac{(a+h-x)^{n-1}}{\lfloor n-1}f^n(x)$$
$$-Ap(a+h-x)^{p-1}$$

putting $x=a+\theta h$ we get

$$g'(a+\theta h)=\frac{h^n(1-\theta)^{n-1}}{\lfloor n-1}f^n(a+\theta h)-Ah^{p-1}p(1-\theta)^{p-1}$$

or $$0=\frac{h^{n-1}(1-\theta)^{n-2}}{\lfloor n-1}f^n(a+\theta h)-\cancel{Ah^{p-1}}\,Ah^{p-1}p(1-\theta)^{p-1}$$ using (iv) or $A h^{p-1} p(1-\theta)^{p-1} = \dfrac{h^{n-1}(1-\theta)^{n-1}}{\lfloor n-1} f^n(a+\theta h)$

or $A = \dfrac{h^{n-1}(1-\theta)^{n-1}}{\lfloor(n-1)\, h^{p-1} p(1-\theta)^{p-1}}$

$= \dfrac{h^{n-p}(1-\theta)^{n-p}}{\lfloor n-1\, p} f^n(\theta+h)$

putting the value of A in (III)

we get-

$f(a+h) = f(a) + h f'(a) + \dfrac{h^2}{\lfloor 2} f''(a) + \cdots$

$\cdots + \dfrac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + \dfrac{h(1-\theta)^{n-p}}{\lfloor n-1\, p} f^n(a+\theta h)$

Thus the theorem is proved

NOTE - (1) The Expansion after n series in Taylor's theorem is called Remainder which is usually denoted by $R_n$

$\therefore R_n = \dfrac{h^n(1-\theta)^{n-p}}{\lfloor n-1\, p} f^n(a+\theta h) \quad \theta \in (0,1)$

This Remainder is known as Schlomilch's and Rouche's Remainder

(II) when $p=1$

$R_n = \dfrac{h^n(1-\theta)^{n-1}}{\lfloor n-1} f^n(a+\theta h)$ this is called Cauchy's remainder

(III) When $p=n$

$R_n = \dfrac{h^n}{\lfloor n} f^n(a+\theta h)$ ... is called Lagrange's

In this case the series for $f(a+h)$ is called Taylor's Series infinite form and hence $f(a+h) = f(a) + \dfrac{h f'(a)}{\lfloor 1} + \dfrac{h^2 f''(a)}{\lfloor 2} + \cdots + \dfrac{h^n}{\lfloor n} f^n(a+\theta h)$